Kreweras's Narayana Number Identity Has a Simple Dyck Path Interpretation

DAVID CALLAN

Department of Statistics
University of Wisconsin-Madison
Medical Science Center
1300 University Ave
Madison, WI 53706-1532

callan@stat.wisc.edu

arXiv:1203.3999v1 [math.CO] 18 Mar 2012

Abstract

We show that an identity of Kreweras for the Narayana numbers counts Dyck paths with a given number of peaks by number of peak plateaus, where a peak plateau is a run of consecutive peaks that is immediately preceded by an upstep and followed by a downstep.

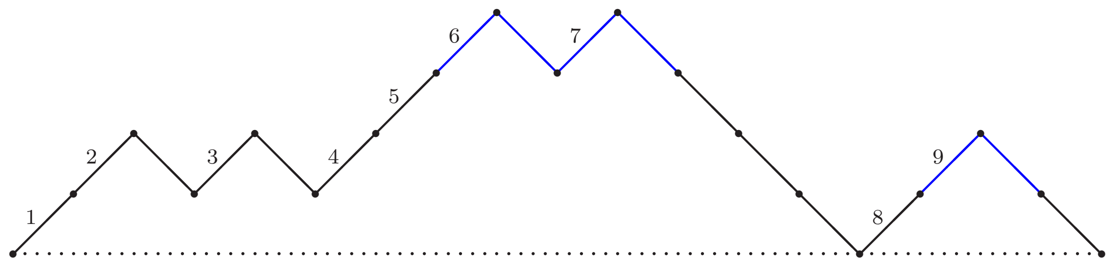
Germain Kreweras [1, 2] proved that

$$\frac{1}{n+r+1}\binom{n+r+1}{r}\binom{n+r+1}{n} = \sum_{s=0}^{r} \frac{1}{n}\binom{n}{r-s}\binom{n}{r-s+1}\binom{2n+s}{2n}.$$

Replacing r by $r-1$, this identity can be rewritten as

$$N(n+r,r) = \sum_{s=1}^{r} N(n,s)\binom{2n+r-s}{r-s} \qquad (*)$$

where $N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the Narayana number, well known to count Dyck n-paths containing k peaks [3, Ex. 6.36, p. 237]. A Dyck n-path is a path of n upsteps U and n downsteps D that never dips below the line joining its start and end points. A peak is an occurrence of UD. We will show that $(*)$ counts Dyck $(n+r)$-paths with r peaks by number s of peak plateaus, where a peak plateau is a run of consecutive peaks that is preceded by an upstep and followed by a downstep, that is, a subpath of the form $U(UD)^iD$ with $i \geq 1$. For example, the path illustrated below has 5 peaks but only 2 peak plateaus (in blue, following upsteps 5 and 8 respectively).



Dyck path with 2 peak plateaus

Given a Dyck $(n+r)$-path with r peaks, delete all its peaks. The result is a Dyck n-path with s peaks where s is the number of peak plateaus in the original path. Each Dyck n-path with s peaks is hit $\binom{2n+r-s}{r-s}$ times: the preimages are found by inserting $r-s$ UDs, repetition allowed, among the $2n+1$ vertices of the n-path—$\binom{2n+r-s}{r-s}$ choices—and then inserting a UD at each of the s original peak vertices, and $(*)$ follows.

References

[1] G. Kreweras, Traitement simultané du "Problème de Young" et du "Problème de Simon Newcomb", *Cahiers du Bur. Univ. de Rech. Opér.* **10**, 1967, 23–31.

[2] T. V. Narayana, *Lattice Path Combinatorics With Statistical Applications*, Mathematical Expositions No. 23, Univ. of Toronto Press, 1979.

[3] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/ .